Improving Secondary Mathematics:

What is Needed?

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Logistics

- Slides will be available at
  - http://bradfindell.com
- Most examples are based on the Common Core State Standards for Mathematics (CCSSM)
  - But the ideas are broadly applicable.

Underlying Principle

- *"Everyone is good at mathematics because everyone can think. And mathematics is about thinking."*
  - Yeap Ban Har, National Institute of Education, Singapore.

- Corollary 1: Strategies that attempt to remove thinking from learning are bound to fail in the long run.
- Corollary 2: When learning is effective, "getting the right answer" is but a small piece of the work.
- Corollary 3: The most important thing teachers and parents can do is convince students that their thinking matters.

Opportunities and Challenges

for improving high school mathematics

Focus, Coherence, and Rigor

- Is there too much content in the CCSSM for high school?
  - NCTM's *Curriculum Focal Points* (2006) was grades K-8
- Integrated or traditional courses?
  - Some standards cut across courses
- Apparent overlap between grade 8 and high school
- The needs of career-intending students are hard to see
  - Though modeling provides good hooks

- CCSSM for high school are not as focused, not as clear, and not as polished as the K-8 standards

Conceptual Bridges

- Meanings of numbers, operations, and place value
  - Multiplication: repeated addition, array and area models
  - Division: How many groups? How many in one group?
  - Partial products
- Proportional relationships between quantities
  - "Write an equation and solve" rather than "set up a proportion and cross multiply"
- Mean absolute deviation
- Rule of four: numerically, symbolically, graphically, and verbally (in context)

- We need additional conceptual bridges for high school content

Collaboration

- Secondary teachers often skim the standards and say,
  - "We teach that."
- Secondary teachers believe they know the content
- Secondary teachers are sometimes worried that they might reveal they don't know something

- What is needed?
  - A safe environment for teachers to work together
  - Course-specific Professional Learning Communities
  - Coaching

Digging into the Mathematics

Content Shifts in Grade 8 and Earlier

- Fluency with standard algorithms, supported by strategies based in place value and properties of operations
- Fractions as numbers on the number line, beginning with unit fractions
- Area models for multiplication
- Much algebra, geometry, and statistics in grade 6-8
- Proportional relationships
  - Unit rates, graphs, tables, formulas, contexts
  - Compare with non-proportional relationships
- Using properties of operations to explain operations with rational numbers

High School Content Shifts

- Number and quantity
  - Number systems, attention to units
- Modeling
  - Threaded throughout the standards
- Geometry
  - Proof for all, transformations
- Algebra and functions
  - Organized by mathematical practices
- Statistics and probability
  - Inference for all, based on simulation

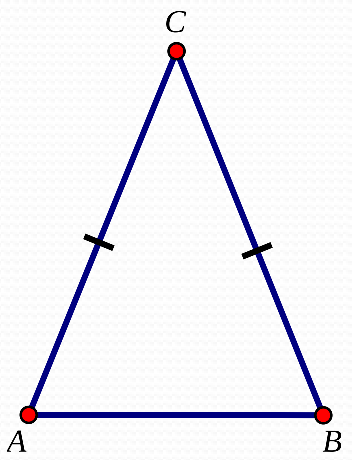
Standard Misunderstandings

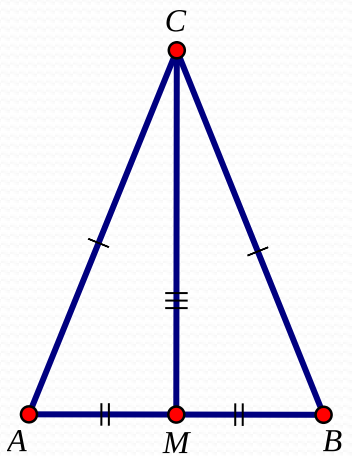
- Complex numbers
  - Division not required
- Inverses of functions
  - Very modest expectations: solving equations $f(x) = c$.
- Logarithms
  - Very modest expectations: A shorthand solution to $b^x = k$.
- Trigonometry
  - Just enough to model periodic phenomena
- Sequences and series
  - Need not be taught together
  - Formulas not needed
- Proof
  - Not two columns

Isosceles Triangle Theorem

- Given $AC = BC$, prove $m\angle A = m\angle B$.

Statement	Reason
1. $AC = BC$	1. Given.
2. $\overline{AC} \cong \overline{BC}$	2. Definition of congruence.
3. Let M be midpoint of $\overline{AB}$	3. Existence of midpoint.
4. $\overline{AM} \cong \overline{BM}$	4. Definition of midpoint.
5. Draw $\overline{CM}$	5. Two points determine a line.
6. $\overline{CM}$ is a median	6. Definition of median.
7. $\overline{CM} \cong \overline{CM}$	7. Reflexive property of congruence.
8. $\Delta AMC \cong \Delta BMC$	8. SSS.
9. $\angle A \cong \angle B$	9. CPCTC.
10. $m\angle A = m\angle B$	10. Definition of congruence.

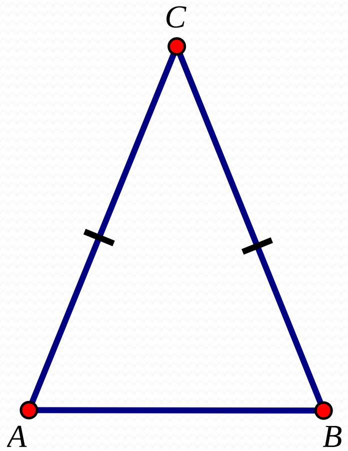


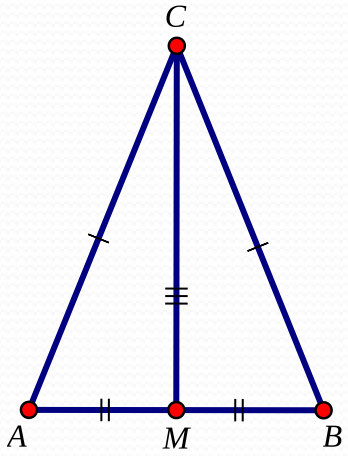


Isosceles Triangle Theorem

- Given $AC = BC$, prove $m\angle A = m\angle B$.

- Draw median $\overline{CM}$ to side $\overline{AB}$.
- Then, as shown in marked figure, $\Delta AMC \cong \Delta BMC$ by SSS.
- So $m\angle A = m\angle B$ from the congruent triangles.





Medians of ≅ Triangles are ≅

Given: $\triangle ABC \cong \triangle EFG$
$\overline{CD}$ is a median of $\triangle ABC$
$\overline{GH}$ is a median of $\triangle EFG$

Prove: $\overline{CD} \cong \overline{GH}$

Statements	Reason
1. $\triangle ABC \cong \triangle EFG$	1. Given
2. $\overline{AC} \cong \overline{EG}$	2. CPCTC
3. $\angle A \cong \angle E$	3. CPCTC
4. $\overline{AB} \cong \overline{EF}$	4. CPCTC
5. $\overline{CD}$ is a median of $\triangle ABC$	5. Given
6.	6. Definition of a median
7. $\overline{AD} \cong \overline{DB}$	7.
8. $AD = \frac{1}{2}AB$	8. Division property of equality
9. $\overline{GH}$ is a median of $\triangle EFG$	9. Given
10.	10. Definition of a median
11. $\overline{EH} \cong \overline{HF}$	11.
12. $EH = \frac{1}{2}EF$	12. Division property of equality
13. $\frac{1}{2}AB = \frac{1}{2}EF$	13. Multiplication property of equality (see #4)
14. $AD = EH$	14.
15. $\overline{AD} \cong \overline{EH}$	15. Definition of congruence
16. $\triangle ADC \cong \triangle EHG$	16.
17. $\overline{CD} \cong \overline{GH}$	17.

∴ Medians of congruent triangles are congruent.

Medians of ≅ Triangles are ≅

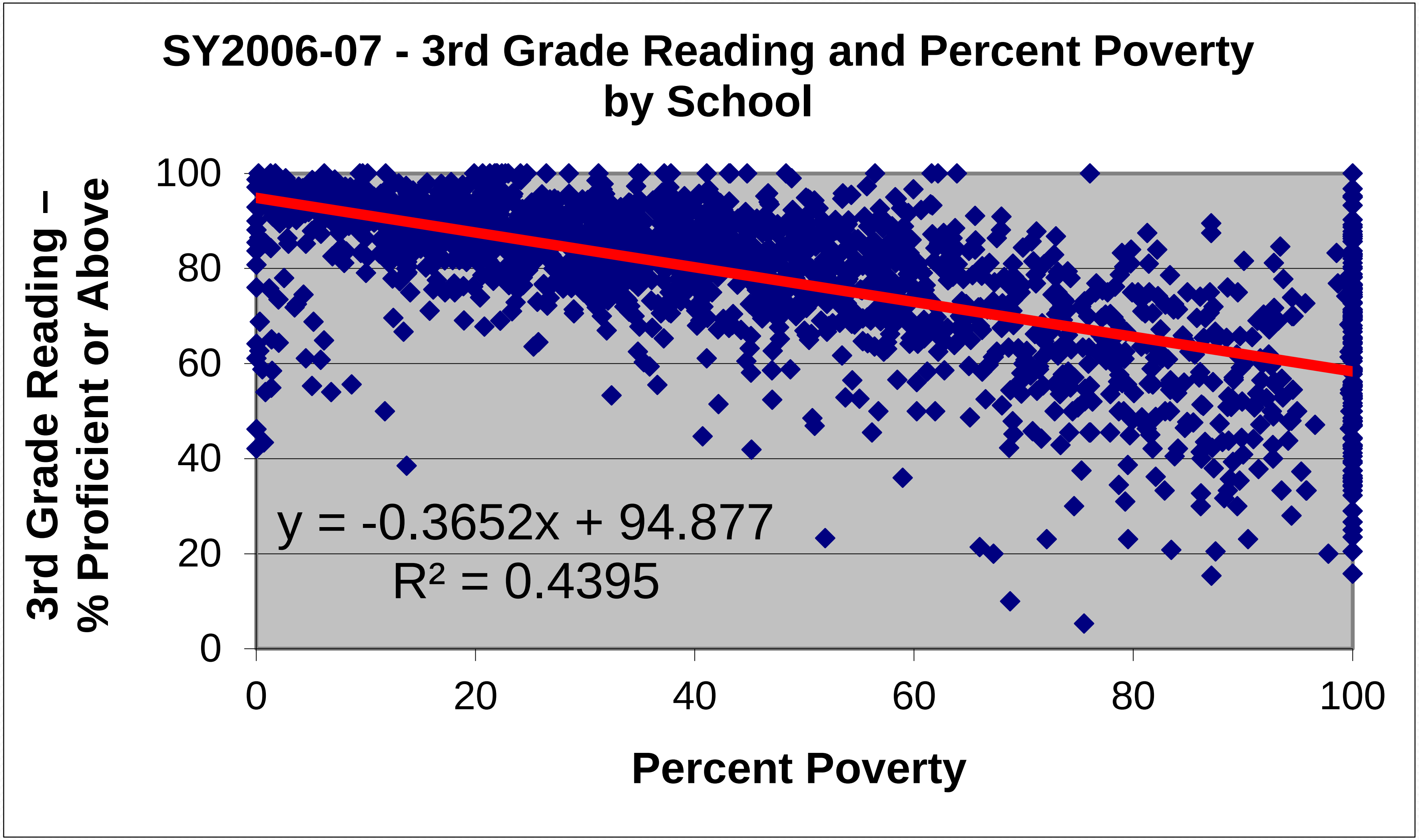
- Proof 1:
  - Duh!
- Proof 2:
  - Because the triangles are congruent, there is a sequence of basic rigid motions that maps one triangle onto the other.
  - Because rigid motions preserve lengths, the midpoints and therefore the medians will coincide, and thus they are congruent.

Two Column Proofs

- Obscure the key ideas
  - Every step carries the same weight
- Requires remembered or invented names for every possible "reason"
- Give the impression that mathematics is about minutia
- Restrict access to few students

- *Let's put two-column proofs to rest*

Who Can Interpret This?



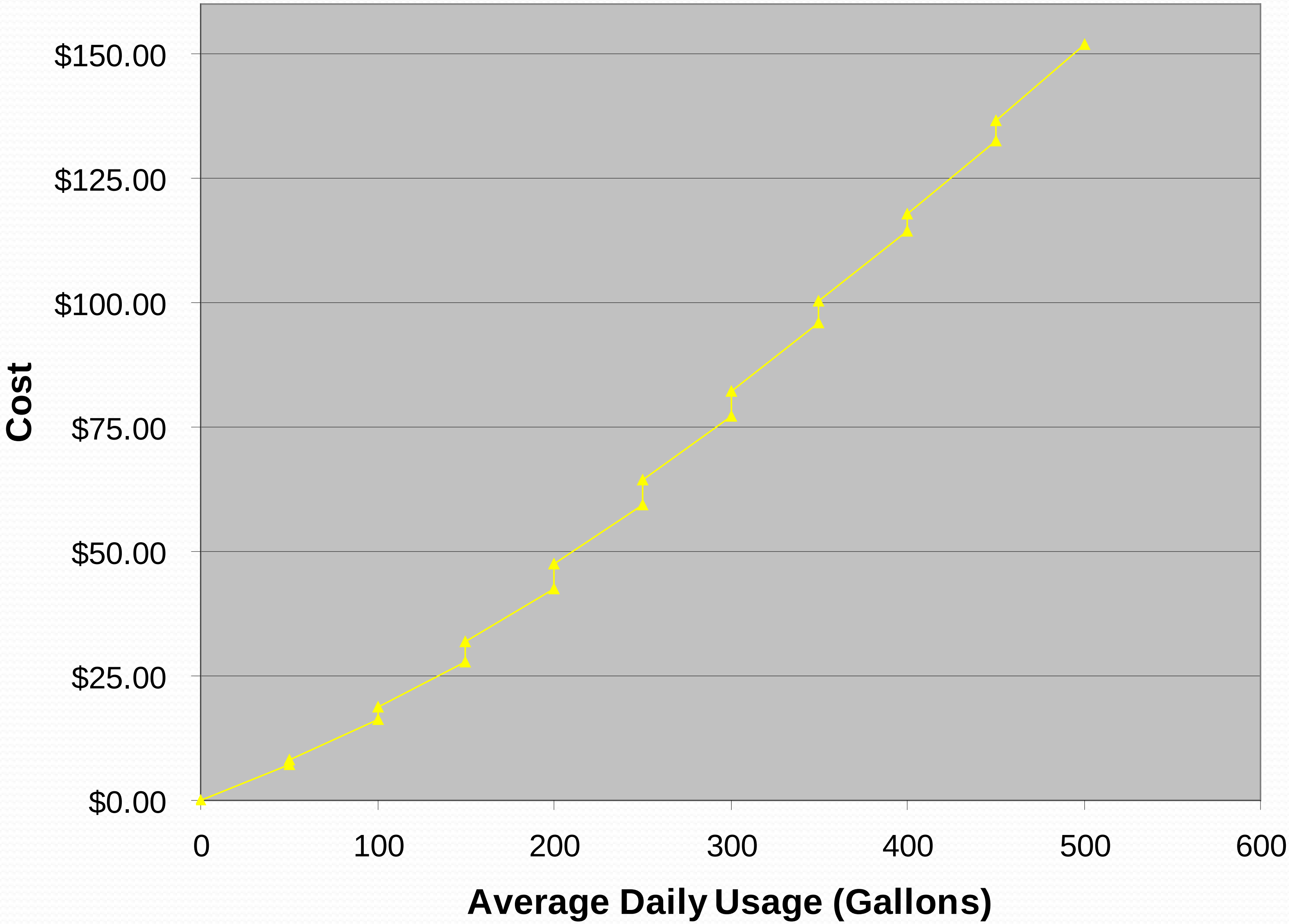
Washington Suburban Sanitary Commission

Rate Schedule, July 1, 2008

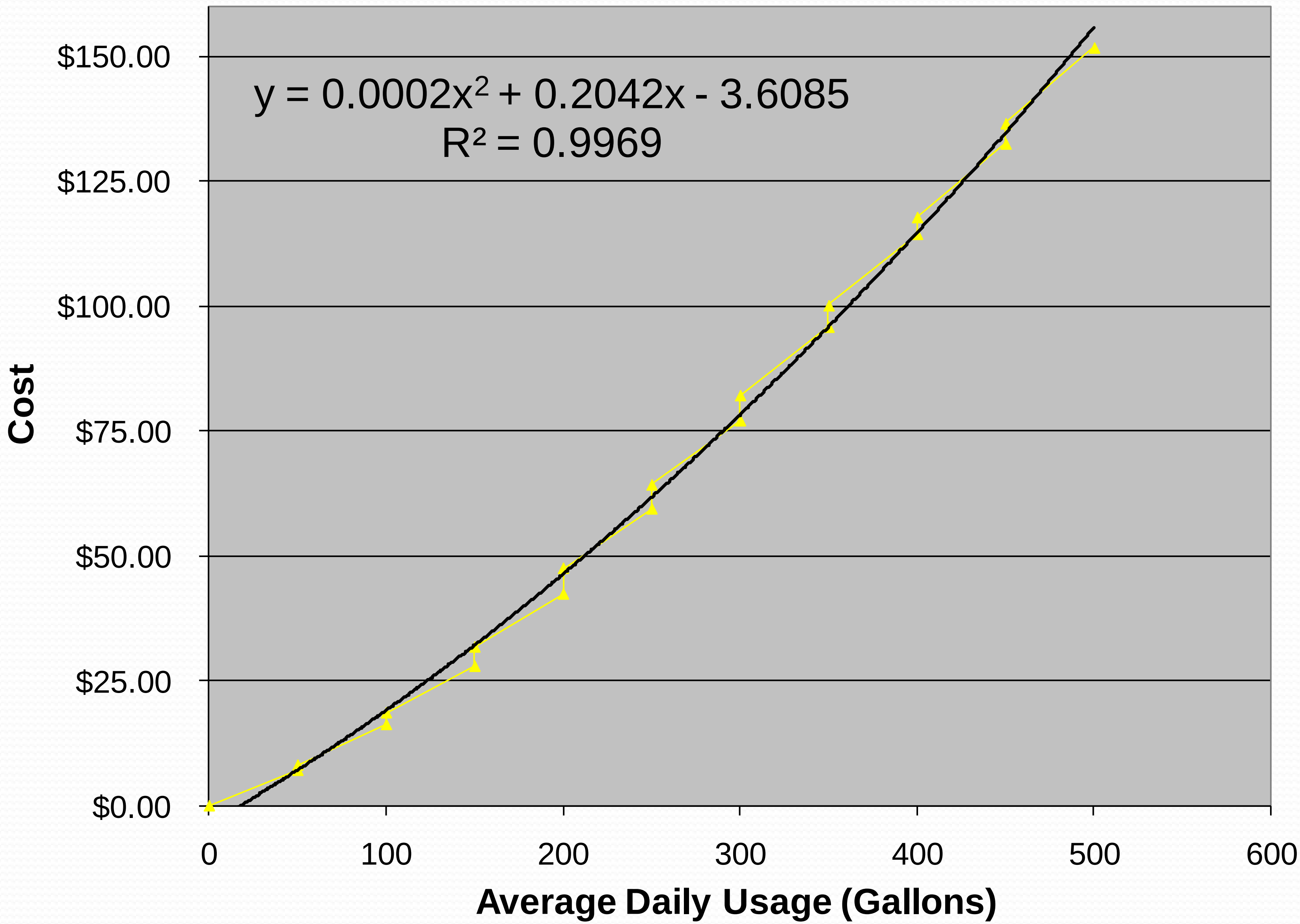
Average Daily Consumption (Gallons/Day)	Water Rate Per 1,000 Gallons	Sewer Rate Per 1,000 Gallons	Combined Rate Per 1,000 Gallons
0-49	$1.97	$2.77	$4.74
50 - 99	2.21	3.22	5.43
100 - 149	2.42	3.79	6.21
150 - 199	2.71	4.36	7.07
200 - 249	3.17	4.76	7.93
250 - 299	3.43	5.14	8.57
300 - 349	3.63	5.50	9.13
350 - 399	3.79	5.75	9.54
400 - 449	3.94	5.88	9.82
…	…	…	…

Source: http://www.wsscwater.com/service/rates.cfm

Monthly Water and Sewer Cost



Monthly Water and Sewer Cost

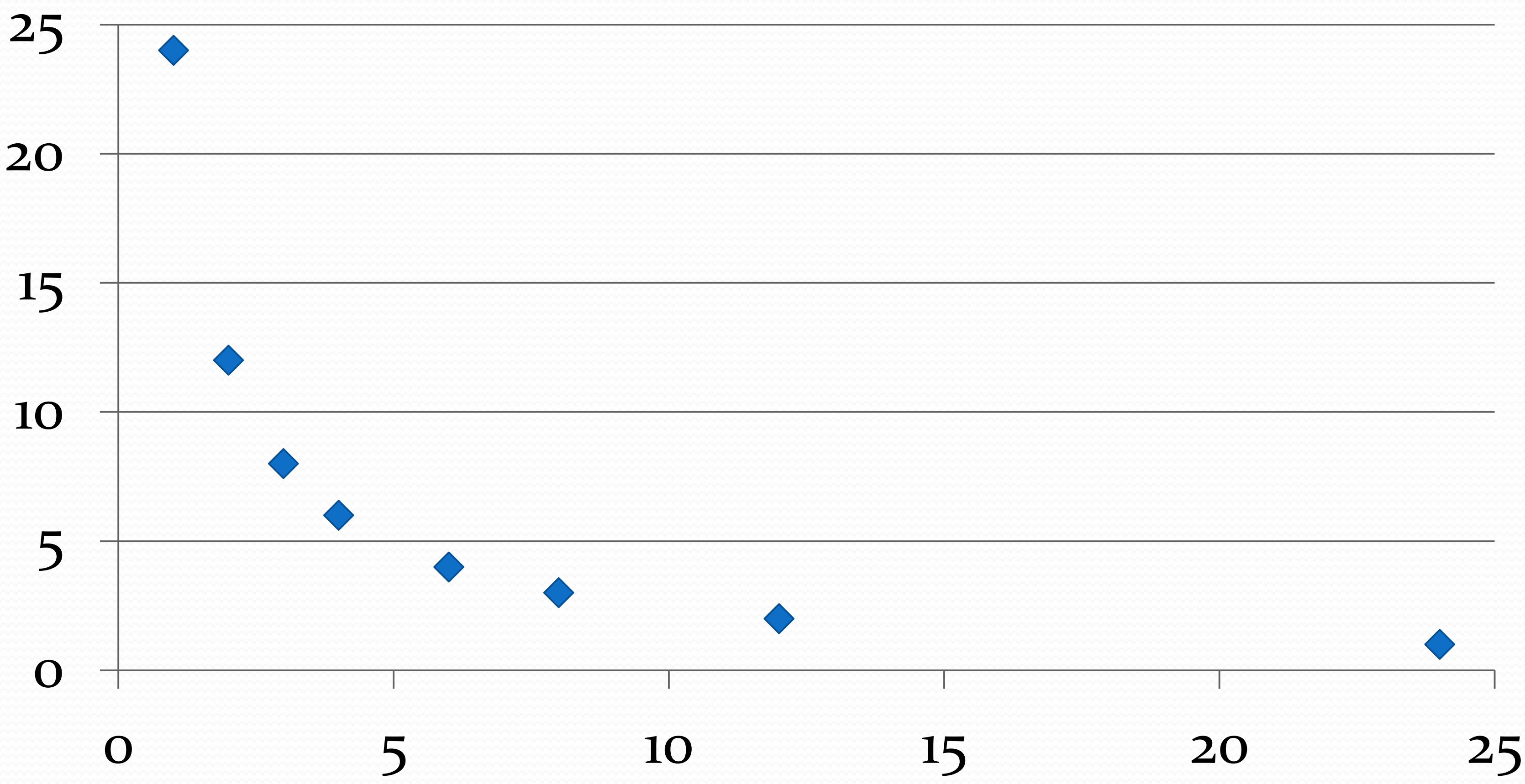


Task Progression

Constant Area, Changing Perimeter

- You have been asked to put together the dance floor for your sister's wedding. The dance floor is made up of 24 square tiles that measure one meter on each side.
  - Experiment with different rectangles that could be made using all of these tiles
  - Record your data in a table and a graph
  - Look for patterns in the data

Width vs. Length

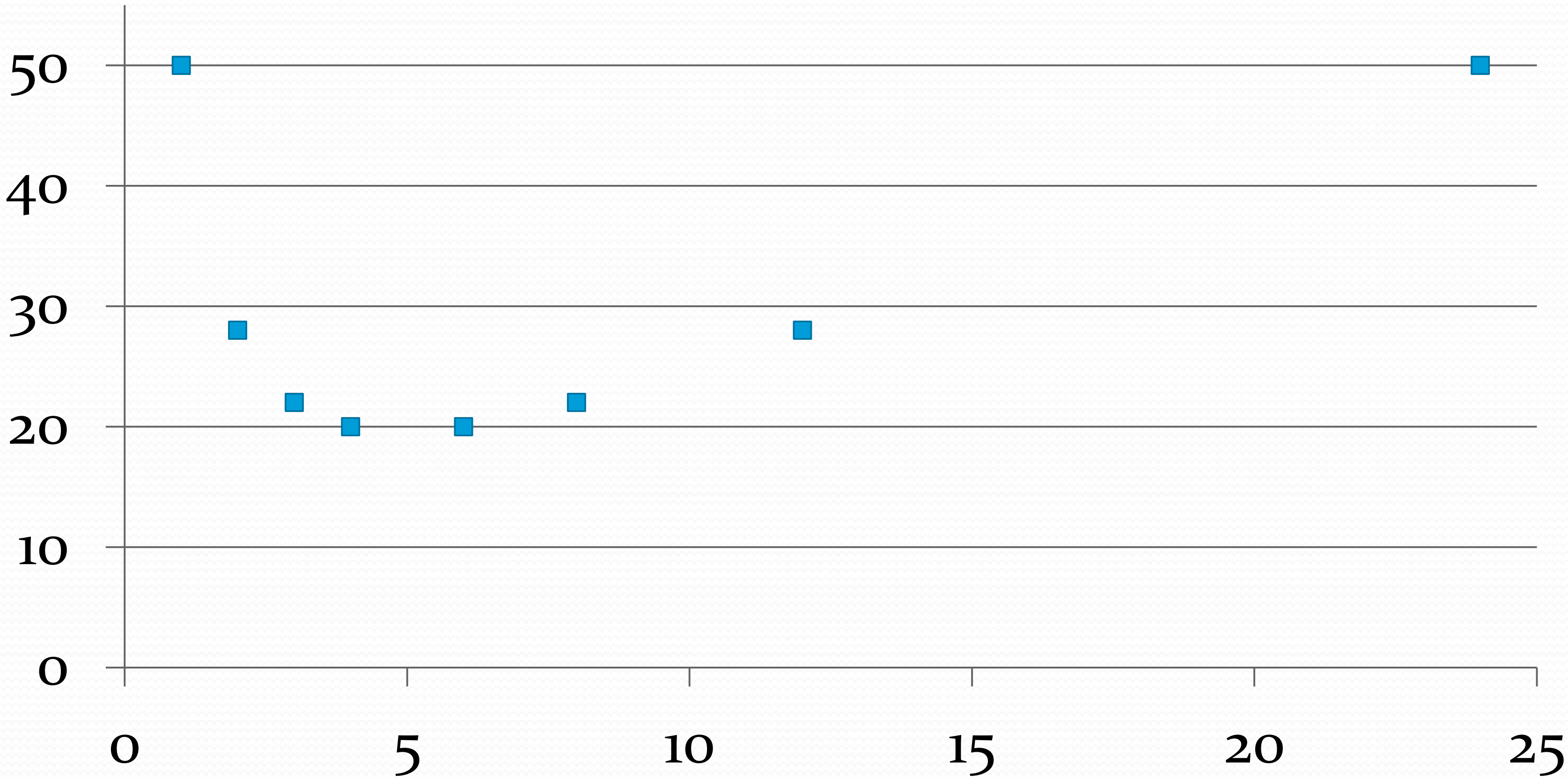


- Suppose the dance floor is held together by a border made of edge pieces one meter long.
  - What determines how many edge pieces are needed: area or perimeter? Explain.

Perimeter vs. Length

- Make a graph showing the perimeter vs. length for various rectangles with an area of 24 square meters.
- Describe the graph. How do patterns that you observed in the table show up in the graph?
- Which design would require the most edge pieces? Explain.
- Which design would require the fewest edge pieces? Explain.

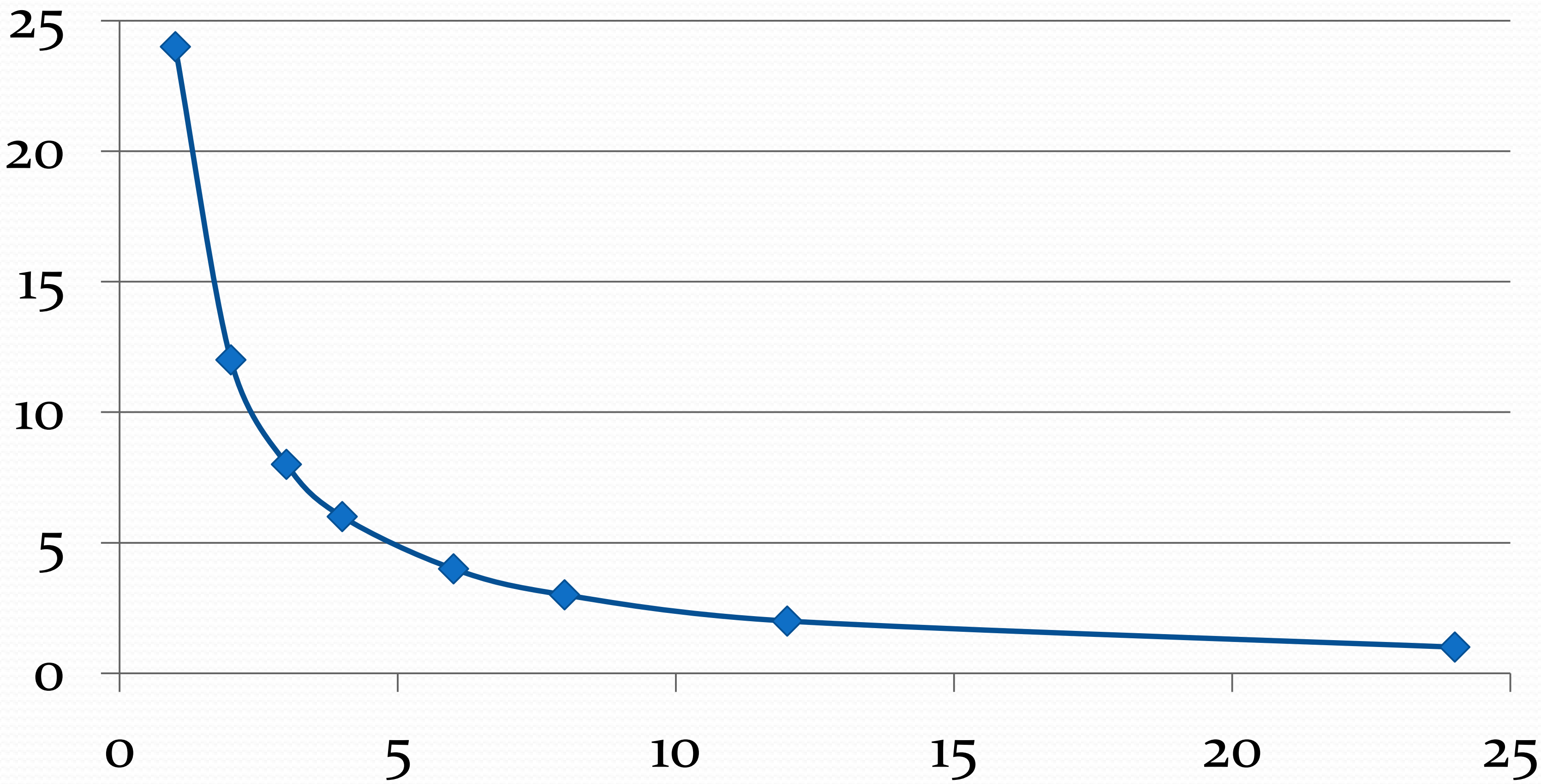
Perimeter vs. Length



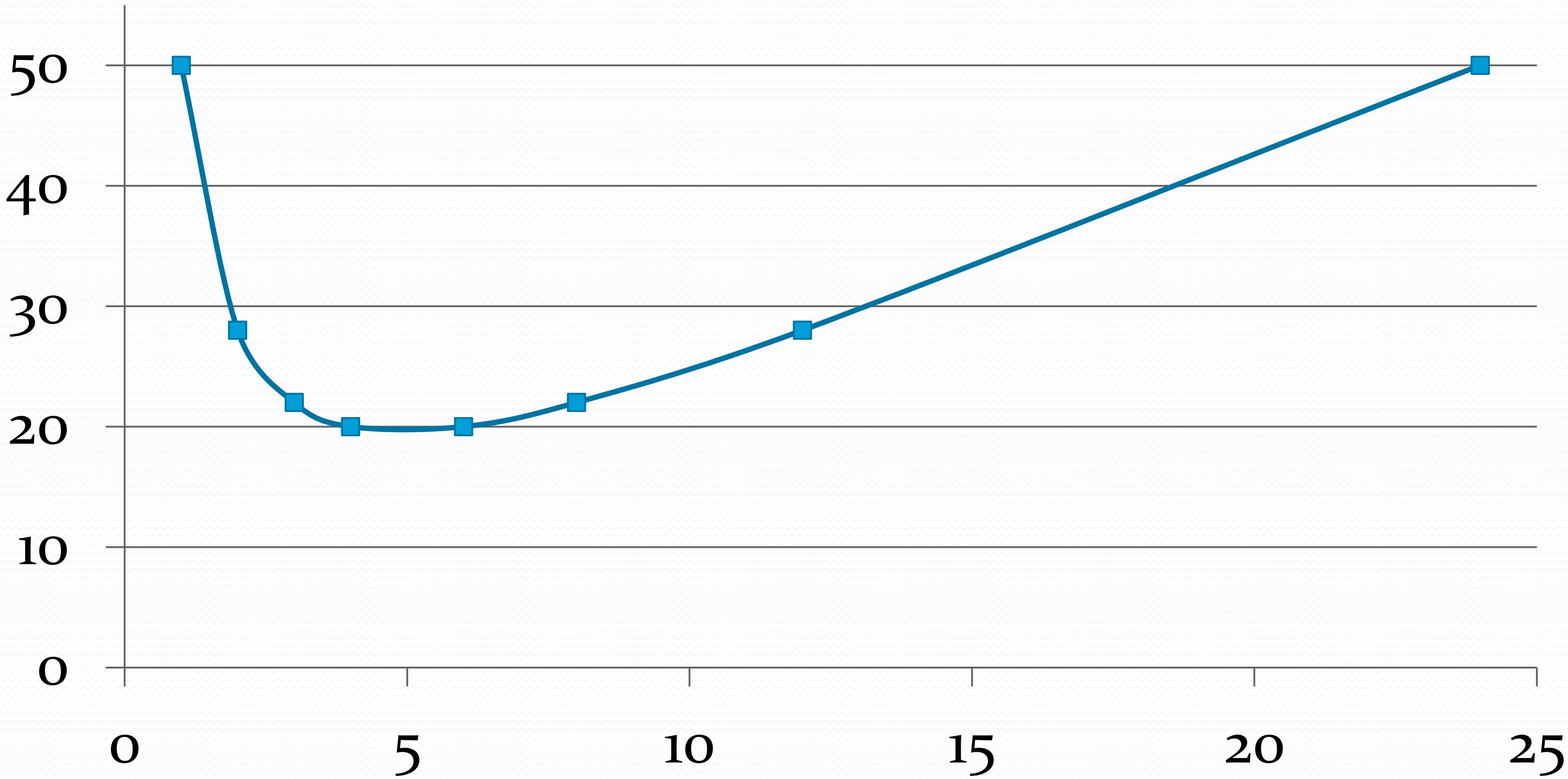
Extension Questions

- Can we connect the dots? Explain.
- How might we change the context so that the dimensions can be other than whole numbers?
- How would the previous answers change?
- In general, describe the rectangle with whole-number dimensions that has the greatest perimeter for a fixed area. Which rectangle has the least perimeter for a fixed area?

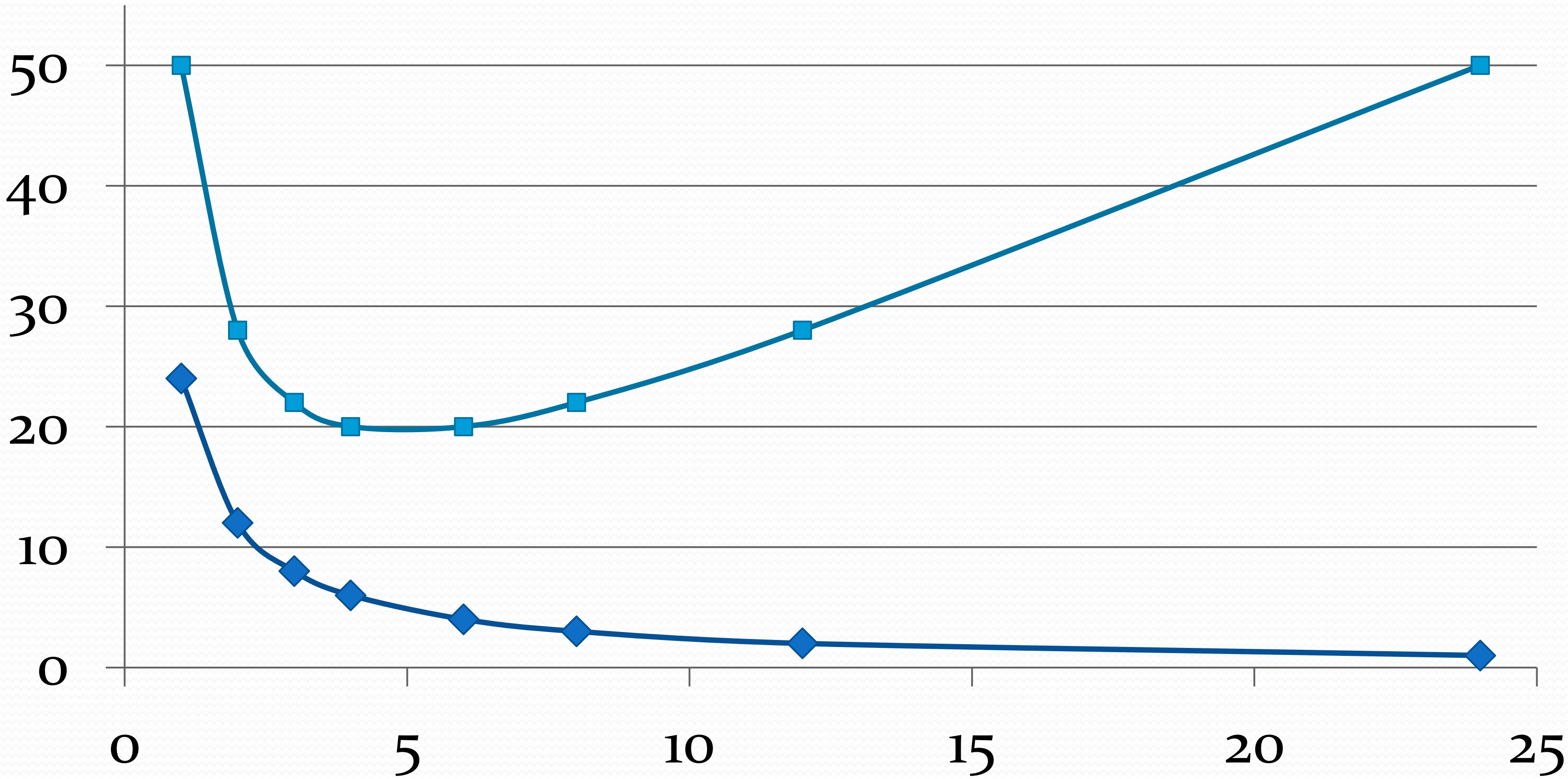
Width vs. Length



Perimeter vs. Length



Perimeter and Width vs. Length



Related Problems

- What if we fix the perimeter?
  - Explore width vs. length
  - Explore area vs. length
- What if we fix the width?
  - Explore area vs. length
  - Explore perimeter vs. length
- Explore these functions in Geogebra
  - http://tube.geogebra.org/material/show/id/978091
  - What kinds of functions are these?
    - Explain graphically, symbolically, in tables, and in context

Questions for Teachers

- How might we use this context to support the learning at the level of Algebra 2 or its equivalent?
  - Domain and range
  - Limiting cases
  - Intercepts and asymptotic behavior?
  - Rates of change, maxima and minima
  - Equation solving with several variables?
  - Generalizing from a specific to a generic fixed quantity?

Perimeter and Area of Rectangles

- Fix one and vary the other
  - Grade 3: to distinguish the two quantities
  - Grade 5: to plot ordered pairs to see relationships
  - Grade 8: to represent the quantities algebraically and to use graphs, tables, and formulas to explore how they are related
  - Grade 11: to distinguish linear, quadratic, and rational functions, and to explore domains in context and to push toward limiting cases
  - Calculus: as an optimization context in which to use differentiation
- Later, in multivariable calculus, explore relations among 3 or more variables

Other Qualitative Shifts in Content

Pythagorean Theorem

- 8.G.6. Explain a proof of the Pythagorean theorem and its converse.
- 8.G.8. Apply the Pythagorean Theorem to find the distance between two points in a coordinate system.
- G-SRT.4. Prove theorems about triangles. *Theorems include … the Pythagorean Theorem proved using triangle similarity.*
- G-SRT.8. Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems.
- G-GPE.1. Derive the equation of a circle of given center and radius using the Pythagorean Theorem …
- F-TF.8. Prove the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$ and use it to calculate trigonometric ratios.

- *The Pythagorean Theorem is not just "$a^2 + b^2 = c^2$."*

Sequences as Functions

- F-IF.3. Recognize that sequences are functions, sometimes defined recursively, whose domain is a subset of the integers.
- F-BF.2. Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms.★
- F-LE.2. Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (include reading these from a table).

- *Emphasize connections among patterns, sequences, and functions.*

Rules of Exponents

- 8.EE.1. Know and apply the properties of integer exponents to generate equivalent numerical expressions.
- N-RN.1. Explain how the definition of the meaning of rational exponents follows from extending the properties of integer exponents to those values, allowing for a notation for radicals in terms of rational exponents.

- *These ideas support many HS standards on exponential functions.*

Solving Equations

- 8.EE.7.a. Give examples of linear equations in one variable with one solution, infinitely many solutions, or no solutions. Show which of these possibilities is the case by successively transforming the given equation into simpler forms, until an equivalent equation of the form $x = a$, $a = a$, or $a = b$ results (where a and b are different numbers).
- A-REI.A. Understand solving equations as a process of reasoning and explain the reasoning
- A-REI.11. Explain why the x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$;

- *This last standard supports the many different techniques for solving different types of equations.*

Seeing Structure in Expressions

- A-SSE.1. Interpret expressions that represent a quantity in terms of its context.
- A-SSE.2. Use the structure of an expression to identify ways to rewrite it.
- A-SSE.3. Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.

- *"Simplest form" depends on the purpose.*

Rational and Irrational Numbers

- 7.NS.2.d. Convert a rational number to a decimal using long division; know that the decimal form of a rational number terminates in 0s or eventually repeats.
- 8.NS.1. Understand informally that every number has a decimal expansion; the rational numbers are those with decimal expansions that terminate in 0s or eventually repeat. Know that other numbers are called irrational.
- N-RN.3. Explain why the sum or product of two rational numbers is rational; that the sum of a rational number and an irrational number is irrational; and that the product of a nonzero rational number and an irrational number is irrational.

- *This content is seldom taught.*

Can You Provide Three Explanations?

- Why is division by 0 undefined? Is 0/0 = 1?
- Why is $a^0 = 1$? And does it matter what *a* is?
- Why is $a^{-n} = 1/a^n$? And does it matter what *a* is?
- Is 0 even, odd, neither, or both?
- Why is a negative times a negative positive?
- When multiplying fractions, why do we multiply numerators and denominators?
- When dividing by a fraction, why is it okay to invert and multiply?
- Is 0.9999… = 1?

Toward Focus and Coherence

Key Ideas for College and Career Readiness

- Rates of change
- Modeling
  - Direct and inverse proportions
  - Linear, quadratic, and exponential functions
- General thinking
  - Thinking generally with specific numbers
  - Seeing specific examples in general statements
  - Seeing structure in expressions: recognizing the "form" of an expression
- Rule of four: numerically, symbolically, graphically, and verbally (in context)

Key Ideas for College and Career Readiness

- Distinguishing expressions, equations, and functions
  - Expressions have values
  - Equations have solutions
  - …
- Meaning of notation
  - order of operations, function notation, absolute value, exponents, fraction, …
  - Mathematician's law of the repeated variable
- Important connections
  - Exponential functions, exponents, radicals
  - Number of solutions

Reminders

- What has been called "Algebra 1" begins in Grade 8 for all students
  - High school Algebra should build on Grade 8
- Universities want students using algebra every year
  - Geometry courses cannot be a "year off" from algebra

- This is important work! Let's work together.