Reasoning and Explaining in the Mathematics Classroom, Grade 6-12: What Counts as Proof?

Some Examples, Non-Examples, and Almost Examples

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Opportunities for Explaining

- 3.OA.9. Identify arithmetic patterns … and explain them using properties of operations.
- 3.MD.7. [Explain] commutativity of multiplication
- 4.NF.1. Explain equivalence of fractions using visual fraction models)
- 4.NBT.5. [Explain] multiplication of two two-digit numbers using strategies based on place value and the properties of operations.
- 6.G.1. [Explain] areas of polygons.
- 8.EE.1. Explain properties of integer exponents.
- 8.EE.6. Explain why slope is the same between any two points on a line
- 8.EE.7. Explain an equation with no solutions or infinitely many solutions
- 8.NS.1. [Explain] informally for rational numbers why their decimals terminate or repeat.

Opportunities for Explaining

- F.IF.8. Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.
- A.REI.11. Explain why the x-coordinates of the points where the graphs of the equation $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$
- N.RN.1. Explain how the definition of the meaning of rational exponents follows from extending the properties of integer exponents to those values, allowing for a notation for radicals in terms of rational exponents.
- N.RN.3. Explain why the sum or product of two rational numbers is rational; that the sum of a rational number and an irrational number is irrational; and that the product of a nonzero rational number and an irrational number is irrational.
- G.GPE.5. Justify the slope criteria for parallel and perpendicular lines.
- G.CO.14. Classify two-dimensional figures in a hierarchy based on properties.

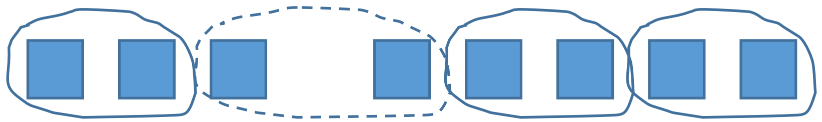
The Sum of Two Odd Numbers Is Even (Proof 1)

- When we go to recess, we line up with partners.
- If one student doesn’t have a partner, that means there are an odd number of students.
- If two classrooms each have an odd number of students, then the two left-over students can be partners with each other.
- So the two classrooms together will have an even number of students.

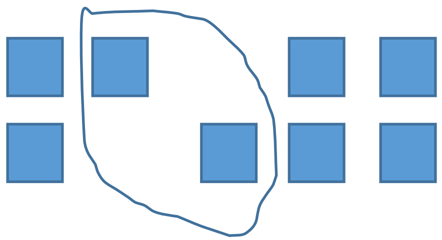
- *Some kindergarteners can produce the gist of this proof.*

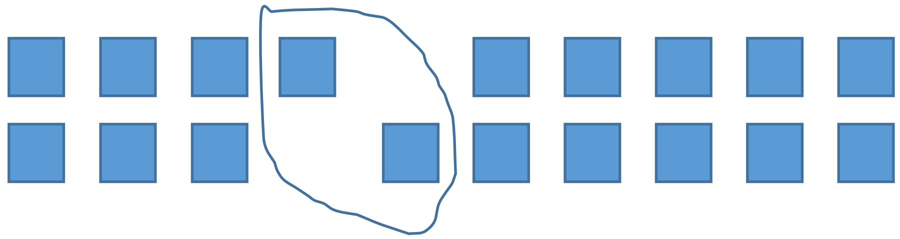
The Sum of Two Odd Numbers is Even (Proof 2)

- True. For example, 3 + 5 = 8. And 7 + 11 = 18. Here is a picture:

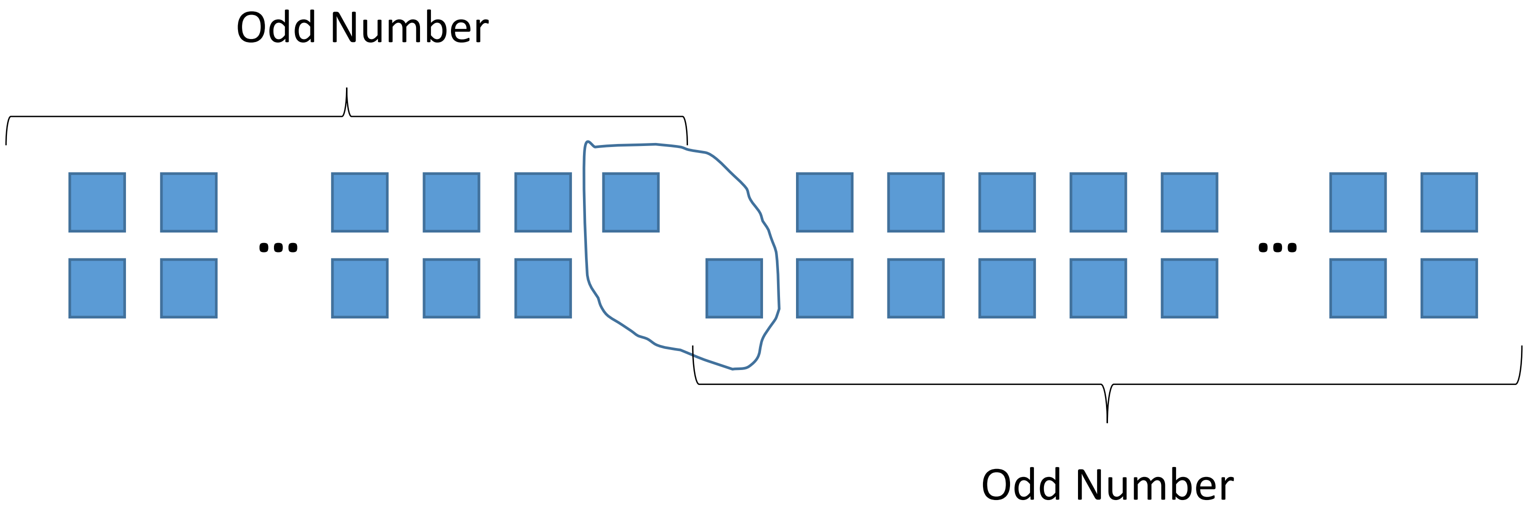


- These kinds of "proofs" disregard the nature of proof—the generality that a proof should convey.
- But in what way are the following pictures better?





The Sum of Two Odd Numbers is Even (Proof 3)



- The structure of the previous pictures suggested generality that is here represented with ellipses (*aka* dots).
- What words should accompany this to make it a proof?

The Sum of Two Odd Numbers is Even (Proof 4)

- Suppose n and m are two odd numbers.
- Then $n = 2k + 1$ and $m = 2j + 1$ for some integers j and k.
- Then $n + m = (2k + 1) + (2j + 1) = 2 (j + k + 1)$, which is an even number.

- Questions:
  - How is this different from the kindergarten proof? From the pictorial proof?
  - Would this get full credit?
  - What would this look like in two columns?

The Sum of Two Odd Numbers Is Even (Proof 5)

Why don’t we expect two-column proofs in algebra?

Statement	Reason
1. Let *n* and *m* be two odd numbers.	Given
2. Then $n = 2k + 1$ and $m = 2j + 1$ for some integers *j* and *k*.	Definition of odd number
3. Then $n + m = (2k + 1) + (2j + 1)$	Substitution
4. Then $n + m = 2k + 1 + 2j + 1$	Associative property of addition
5. Then $n + m = 2k + 2j + 2$	Associative and commutative properties of addition
6. Then $n + m = 2\ (j + k + 1)$	Distributive property of multiplication over addition
7. $B = j + k + 1$ is an integer	Closure property of the integers
8. Then $n + m = 2B$, which is even	Substitution; definition of even number

Note: Previous step 3 became six steps, and still some statements required multiple reasons.

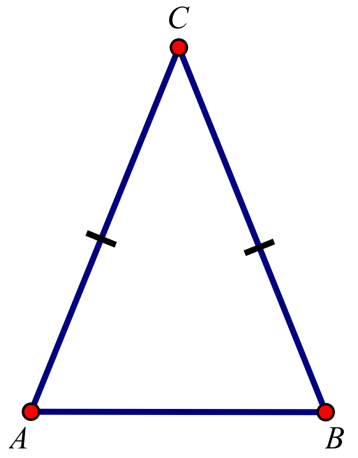
What About These “Proofs”?

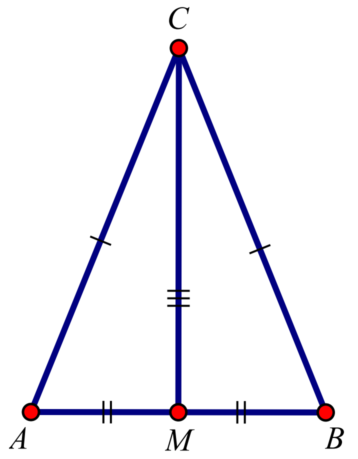
- Proof 6:
  - Suppose n and m are two odd numbers.
  - Then $n = 2k + 1$ and $m = 2j + 1$.
  - Then $n + m = (2k + 1) + (2j + 1) = 2\ (j + k + 1)$, which is even.
- Proof 7:
  - An odd number is $n = 2k + 1$, where k is an integer. The sum of two odd numbers would be $n + n = (2k + 1) + (2k + 1) = 4k + 2 = 2(2k + 1)$, which is even.
- Questions:
  - Is one of these better than the other?
  - How much credit would you give each?

Isosceles Triangle Theorem (Proof 1)

- Given $AC = BC$, prove $m\angle A = m\angle B$.

Statement	Reason
1. $AC = BC$	1. Given.
2. $\overline{AC} \cong \overline{BC}$	2. Definition of congruence.
3. Let M be midpoint of $\overline{AB}$	3. Existence of midpoint.
4. $\overline{AM} \cong \overline{BM}$	4. Definition of midpoint.
5. Draw $\overline{CM}$	5. Two points determine a line.
6. $\overline{CM}$ is a median	6. Definition of median.
7. $\overline{CM} \cong \overline{CM}$	7. Reflexive property of congruence.
8. $\Delta AMC \cong \Delta BMC$	8. SSS.
9. $\angle A \cong \angle B$	9. CPCTC.
10. $m\angle A = m\angle B$	10. Definition of congruence.

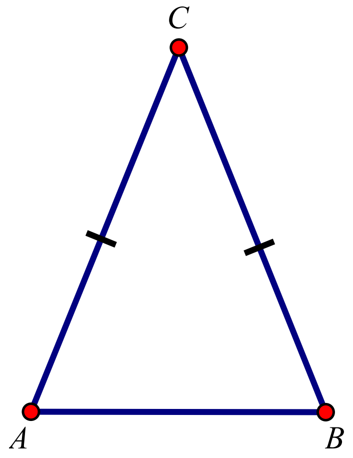


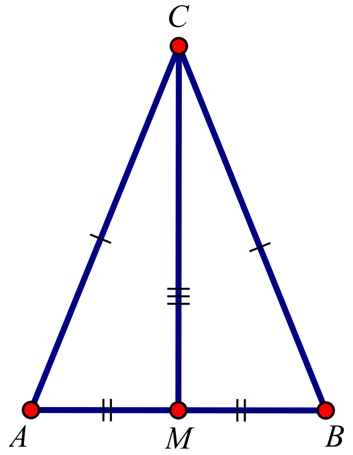


Isosceles Triangle Theorem (Proof 2)

- Given $AC = BC$, prove $m\angle A = m\angle B$.

- Draw median $\overline{CM}$ to side $\overline{AB}$.
- Then, as shown in marked figure, $\Delta AMC \cong \Delta BMC$ by SSS.
- So $m\angle A = m\angle B$ from the congruent triangles.

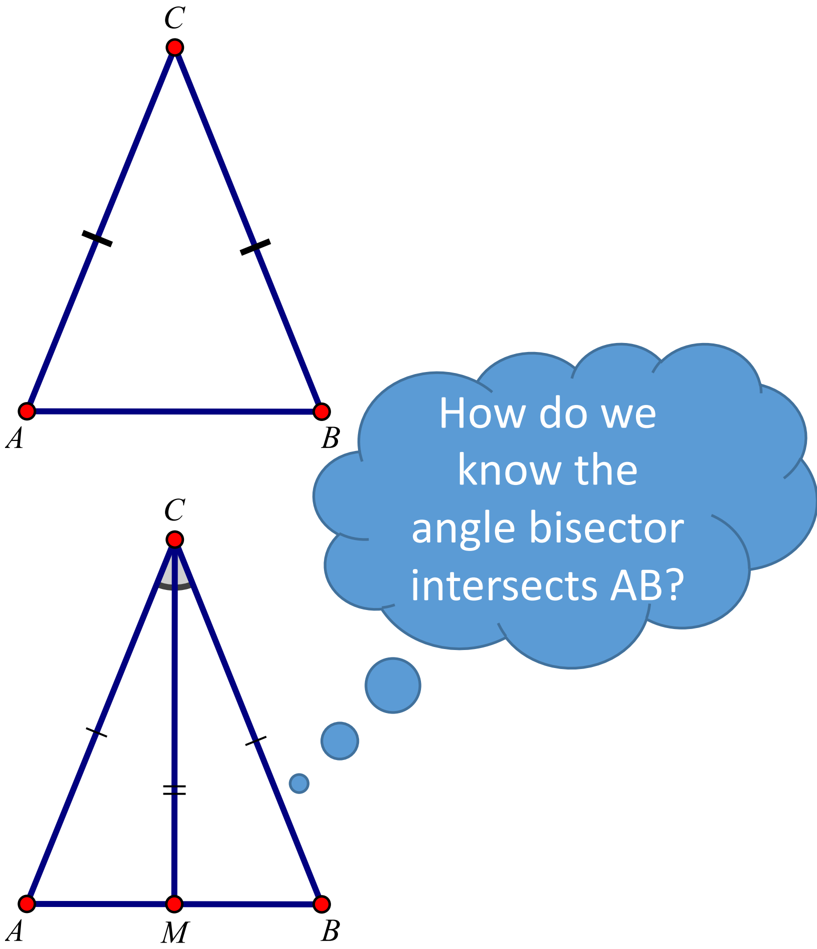




Isosceles Triangle Theorem (Proof 3)

- Given $AC = BC$, prove $m\angle A = m\angle B$.

- Draw angle bisector $\overline{CM}$ to side $\overline{AB}$.
- Then, as shown in marked figure, $\Delta ACM \cong \Delta BCM$ by SAS.
- So $m\angle A = m\angle B$ from the congruent triangles.

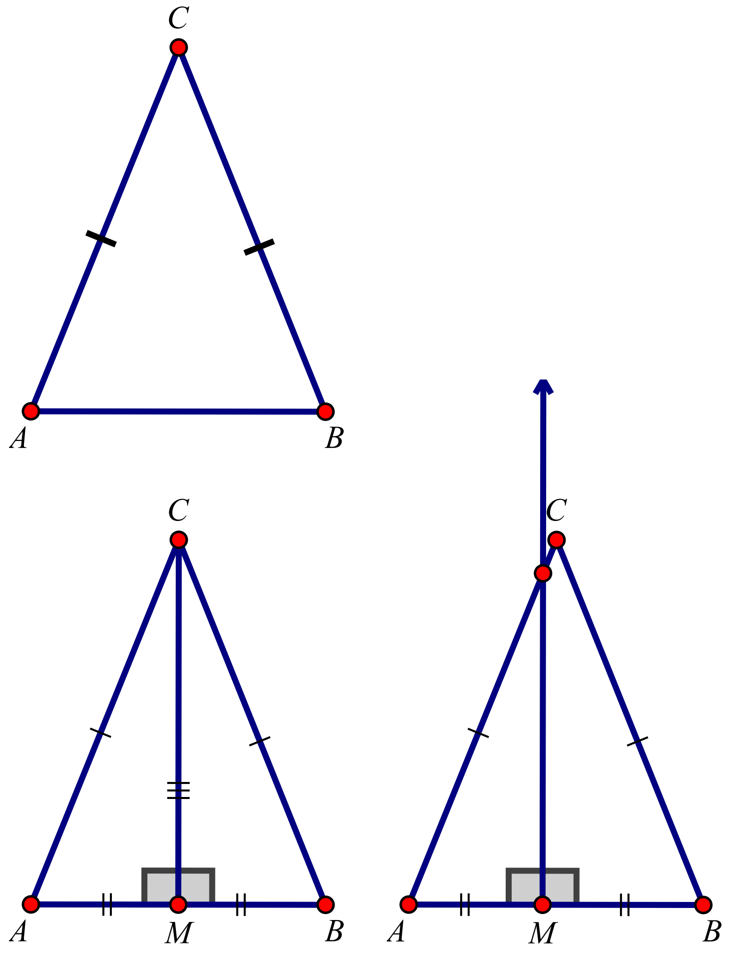


Isosceles Triangle Theorem (Proof 4)

- Given $AC = BC$, prove $m\angle A = m\angle B$.

- Draw perp. bisector $\overline{CM}$ to side $\overline{AB}$.
- Then, as shown in marked figure, $\Delta ACM \cong \Delta BCM$ by …

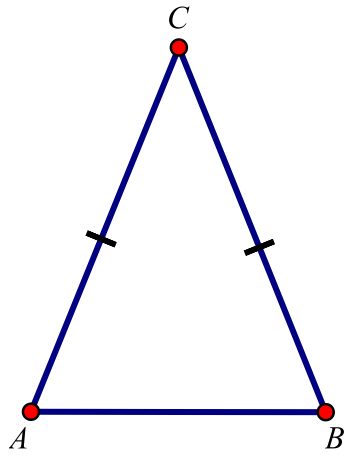
- How do we know the perpendicular bisector contains C?

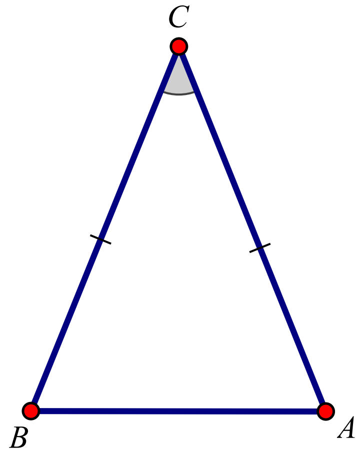


Isosceles Triangle Theorem (Proof 5)

- Given $AC = BC$, prove $m\angle A = m\angle B$.

- $\Delta ACB \cong \Delta BCA$ by SAS.
- So $m\angle A = m\angle B$ from the congruent triangles.





Medians of ≅ Triangles are ≅ (Proof 1)

Given: $\triangle ABC \cong \triangle EFG$
$\overline{CD}$ is a median of $\triangle ABC$
$\overline{GH}$ is a median of $\triangle EFG$

Prove: $\overline{CD} \cong \overline{GH}$

Statements	Reason
1. $\triangle ABC \cong \triangle EFG$	1. Given
2. $\overline{AC} \cong \overline{EG}$	2. CPCTC
3. $\angle A \cong \angle E$	3. CPCTC
4. $\overline{AB} \cong \overline{EF}$	4. CPCTC
5. $\overline{CD}$ is a median of $\triangle ABC$	5. Given
6.	6. Definition of a median
7. $\overline{AD} \cong \overline{DB}$	7.
8. $AD = \frac{1}{2}AB$	8. Division property of equality
9. $\overline{GH}$ is a median of $\triangle EFG$	9. Given
10.	10. Definition of a median
11. $\overline{EH} \cong \overline{HF}$	11.
12. $EH = \frac{1}{2}EF$	12. Division property of equality
13. $\frac{1}{2}AB = \frac{1}{2}EF$	13. Multiplication property of equality (see #4)
14. $AD = EH$	14.
15. $\overline{AD} \cong \overline{EH}$	15. Definition of congruence
16. $\triangle ADC \cong \triangle EHG$	16.
17. $\overline{CD} \cong \overline{GH}$	17.

∴ Medians of congruent triangles are congruent.

Medians of $\cong$ Triangles are $\cong$ (Proof 2)

- Use transformational reasoning:
  - Because the triangles are congruent, there is a sequence of basic rigid motions that maps one triangle onto the other.
  - Because rigid motions preserve lengths, the midpoints and therefore the medians will coincide, and thus they are congruent.

- *These are worthwhile ideas!*

Two-Column Proofs

- Obscure the key ideas
  - Every step carries the same weight
- Require remembered or invented names for every possible “reason”
  - Why do we so seldom use two columns in algebra?
- Give the impression that mathematics is about minutia
  - Sometimes math is about minutia, but let’s save that for undergraduate math majors
- Restrict access to few students
- *Let’s encourage alternative proof formats*
- *Let’s encourage multiple levels of formality*
- *Let’s encourage proof in algebra*

Frege’s Aims of Proof

1. to put the truth of a proposition (the theorem) beyond rational (rather than psychological) doubt;

Understanding

2. to afford insight into the pattern of dependence of truths upon one another;

Explanation

3. to show why a proposition is true; and
4. to reduce the number of basic assumptions underlying mathematics.

- Which of these are most important for high school students?

From http://www.maa.org/press/periodicals/convergence/proofs-without-words-and-beyond-why-we-write-proofs

Improving Precision in Language

- A triangle has 180 degrees.
- A line measures 180 degrees.
- A circle is (or has) 360 degrees.

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- 8.EE.6. Explain why slope is the same between any two points on a line
- 8.EE.7. Explain an equation with no solutions or infinitely many solutions
- 8.G.6. Analyze and justify an informal proof of the Pythagorean Theorem and its converse.
- 8.NS.1. [Explain] informally for rational numbers why their decimals terminate or repeat.

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